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Pseudoradial spaces: Separable subsets, products and maps onto Tychonoff cubes

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Abstract

We work around our question of whether compact non-pseudoradial spaces have separable such subspaces. We obtain results about products of pseudoradial spaces and obtain more conditions which guarantee that each compact sequentially compact space is pseudoradial. We also discuss some questions of Šapirovskiĭ which are also directed at separating the non-separable from the separable. We reinforce the need to focus on the space I^{ω_2}. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

After the work of Šapirovskiĭ on the equivalence (under CH) between sequential compactness and pseudoradiality in compact spaces [8], the pseudoradial spaces have received new attention. Some very recent results can be found for instance in [1,3].

This paper presents further results on the structure of pseudoradial spaces. Our theme might be described as interpreting Šapirovskiĭ's result as suggesting that non-pseudoradiality of compact spaces reflects to the countable. Among other things, we answer a couple of questions of Šapirovskiĭ in this regard.

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We frequently work with Martin's Axiom type assumptions which will take the simple form of asserting equality of some standard cardinal invariants of the continuum. The reader should recall the formula $\omega_1 \leqslant \mathfrak{p} \leqslant \mathfrak{s} \leqslant \mathfrak{i} \leqslant \mathfrak{c}$, for this and more on the small uncountable cardinals see [9,2]. Concerning the notation 2^κ, the context will make clear whether we mean cardinal exponentiation or a topological space.

Henceforth, the word transfinite sequence (briefly sequence) will mean any well ordered net. Radial and pseudoradial spaces are characterized by the degree to which such well-ordered convergence determines the topology.

Let X denote an arbitrary space. A subset A of X is radially (κ-radially) closed if it contains the limit points of all convergent sequences (sequences of size at most κ) consisting of points from A. If each radially closed set is closed, then X is said to be pseudoradial.

The radial character $R\chi(X)$ of the pseudoradial space X is the smallest cardinal number κ such that each κ-radially closed set is closed. Note that a sequential space is just a pseudoradial space with countable radial character. A subset A of the topological space X is ω-closed if $\overline{B} \subseteq A$ for any countable set $B \subseteq A$.

A basic, and still open problem, is to decide if, in ZFC, the class of compact pseudoradial spaces is finitely (or countably) productive.

Since sequential compactness is countably productive, this problem has a consistent solution as an immediate consequence of the following.

Theorem 1.1 [5]. *If $\mathfrak{c} \leqslant \omega_2$ then a compact sequentially compact space is pseudoradial.*

The key point to this is the following theorem

Theorem 1.2 [5]. *If X is compact sequentially compact and A is radially closed but not closed, then there is a cardinal λ, with $\lambda^+ < \mathfrak{c}$, and a closed relative G_λ subset of $\overline{A}$ which misses A.*

As a byproduct of the above theorem, we immediately have that any compact space which is not pseudoradial contains a closed non-pseudoradial subspace Y of density $d(Y) < \mathfrak{c}$.

In fact, this has been further improved in [4], by showing that the continuum $\mathfrak{c}$ can be replaced by the splitting number $\mathfrak{s}$.

2. Separable non-pseudoradial subspaces and products

One can regard Šapirovskiĭ's result as saying something like "if countable subsets are pseudoradial, then a compact space is pseudoradial". This and the results mentioned at the end of the previous section are our motivation for the following:

Problem 1. Suppose that a compact space X is not pseudoradial. Does X have a closed separable subspace which is not pseudoradial?

It is well known that 2^{ω_1} is pseudoradial exactly when $\mathfrak{s}$ is greater than ω_1. However let us recall that in any ccc forcing extension of a model of $2^{\omega_1}=\omega_2$ the space 2^{ω_2} is not pseudoradial. The reason is that ccc forcing preserves the character of the cub filter [6] and if the cub filter on ω_1 has character κ, then 2^κ is not pseudoradial [5].

Theorem 2.1. *If X is a compact space which cannot be mapped onto I^{ω_2} and all closed separable subspaces of X are pseudoradial, then X is pseudoradial.*

Proof. For simplicity, we assume that the space X is zero-dimensional. The proof in the general case is quite a standard modification of the one presented here.

Let us suppose that all closed separable subspaces of X are pseudoradial, but X itself is not pseudoradial. By the discussion at the end of the previous section, we must have $\omega_1<\mathfrak{s}$. Consequently, we also have that the independence number $\mathfrak{i}$ is at least ω_2.

Fix a set $A\subseteq X$ which is radially closed and not closed. Let λ be minimal such that there is a closed set $H\subseteq\overline{A}\setminus A$ which is a G_λ-set in $\overline{A}$. As we are assuming that every closed separable subspace of X is pseudoradial, the space X is actually sequentially compact and so we have that λ has uncountable cofinality. Since A is radially closed, each G_λ-subset of H is infinite.

For this reason we may fix countably many pairwise disjoint relatively clopen subsets of H, $\{H(n,0)\colon n\in\omega\}$. Inductively choose a family $\{B_\mu\colon \mu<\omega_2\}$ of clopen subsets of X and closed sets $\{H(n,\mu)\colon \mu<\omega_2\}$ so that, for each μ, $H(n,\mu+1)$ is set equal to either $H(n,\mu)\cap B_\mu$ or $H(n,\mu)\setminus B_\mu$. If μ is a limit, then we set $H(n,\mu)=\bigcap_{\beta<\mu}H(n,\beta)$. We also choose $\sigma_\mu\in \mathrm{Fn}(\mu,2)$, a finite partial function from μ into 2, such that the following formula $(*_\mu)$ holds

$$\begin{aligned}(*_\mu)\ \ &\text{for all } \tau\in\mathrm{Fn}(\mu,2) \text{ such that } \sigma_\beta=\sigma_\mu \text{ for each } \beta\in\mathrm{dom}(\tau)\\ &\Big(\big|\{n\in\omega\colon H(n,\mu+1)\subseteq B_{\sigma_\mu}\cap(B_\tau\cap B_\mu)\}\big|=\aleph_0 \text{ and}\\ &\big|\{n\in\omega\colon H(n,\mu+1)\subseteq B_{\sigma_\mu}\cap(B_\tau\setminus B_\mu)\}\big|=\aleph_0\Big).\end{aligned}$$

Here, if $\tau\in\mathrm{Fn}(\mu,2)$ we set $B_\tau=\bigcap_{\alpha\in\mathrm{dom}(\tau)}B_\alpha^{\tau(\alpha)}$, where $B_\alpha^{\tau(\alpha)}=B_\alpha$ if $\tau(\alpha)=1$ and $B_\alpha^{\tau(\alpha)}=X\setminus B_\alpha$ if $\tau(\alpha)=0$.

Let us assume that we already have the sets $\{H(n,\mu)\colon n\in\omega\}$ and the sets $\{B_\alpha\colon \alpha<\mu\}$ and let us try to choose B_μ and $H(n,\mu+1)$ for each $n\in\omega$.

Notice that, by our assumptions, for each n and $\alpha<\mu$, $H(n,\mu)$ is either contained in, or disjoint from B_α. For each α, let $Y_\alpha=\{n\in\omega\colon H(n,\mu)\subseteq B_\alpha\}$. Let $\mathcal{Y}_\mu$ be the Boolean subalgebra of $\mathcal{P}(\omega)$ generated by the family $\{Y_\alpha\colon \alpha\in\mu\}$. It follows, trivially, that ω^* maps, by the natural map, onto the Stone space of $\mathcal{Y}_\mu/\mathrm{fin}$. Let $K_\mu\subset\omega^*$ be any closed subset such that when this map is restricted to K_μ, it is irreducible and onto. Let f_μ denote the map from K_μ onto the Stone space of $\mathcal{Y}_\mu/\mathrm{fin}$.

If possible, choose a clopen set $B_\mu\subseteq X$, so that K_μ meets the closure of

$$Z_\mu=\big\{n\colon B_\mu\cap H(n,\mu) \text{ and } H(n,\mu)\setminus B_\mu \text{ are both not empty}\big\}.$$

We will show later that such B_μ does exist for every $\mu<\omega_2$. Now, we describe the way to choose σ_μ. Fix B_μ such that K_μ meets the closure of Z_μ. By the irreducibility of the map

f_μ, $f_\mu[K_\mu \cap \overline{Z_\mu}]$ has interior in the Stone space of $\mathcal{Y}_\mu/\mathrm{fin}$. By the definition of the Stone space topology and the definition of $\mathcal{Y}_\mu/\mathrm{fin}$, there is a $\sigma_\mu \in \mathrm{Fn}(\mu, 2)$ such that this image contains the clopen set, Y'_{σ_μ}, corresponding to the subset of ω, Y_{σ_μ} naturally defined as

$$\bigcap\{Y_\alpha\colon \sigma_\mu(\alpha) = 1\} \cap \bigcap\{\omega \setminus Y_\alpha\colon \sigma_\mu(\alpha) = 0\}.$$

Note that this just means that K_μ is disjoint from the closure of $Y_{\sigma_\mu} \setminus Z_\mu$. It is important to note that it then follows that for all $Y \in \mathcal{Y}_\mu$, such that $Y \cap Y_{\sigma_\mu}$ is infinite, $Y \cap (Y_{\sigma_\mu} \cap Z_\mu)$ is also infinite.

Now, set $J_\mu = \{\beta \in \mu\colon \sigma_\beta = \sigma_\mu\}$. By inductive assumption, $\{Y_\beta \cap Y_{\sigma_\mu}\colon \beta \in J_\mu\}$ is an independent family on Y_{σ_μ}. To see this, take any $\tau \in \mathrm{Fn}(J_\mu, 2)$ and let $\mu' = \max\mathrm{dom}(\tau) < \mu$. Then the formula $|Y_\tau \cap Y_{\sigma_\mu}| = \aleph_0$ follows from the relevant clause of $(*_{\mu'})$ (depending upon the value of $\tau(\mu')$). Now, in addition, $\{Y_\beta \cap (Y_{\sigma_\mu} \cap Z_\mu)\colon \beta \in J_\mu\}$ is an independent family on $Y_{\sigma_\mu} \cap Z_\mu$. However this family is not a maximal independent family on $(Y_{\sigma_\mu} \cap Z_\mu)$, so we can choose an infinite set $Y \subseteq Y_{\sigma_\mu} \cap Z_\mu$ such that $\{Y_\beta\colon \beta \in J_\mu\} \cup \{Y\}$ is still an independent family on $(Y_{\sigma_\mu} \cap Z_\mu)$. For each $n \in Y$, set $H(n, \mu+1) = H(n, \mu) \cap B_\mu$ and, for $n \in Z_\mu \setminus Y$, set $H(n, \mu+1) = H(n, \mu) \setminus B_\mu$. For $n \notin Z_\mu$, let $H(n, \mu+1) = H(n, \mu)$ and redefine B_μ to be $B_\mu \cap B_{\sigma_\mu}$.

Let us verify that the construction cannot stop at some ordinal $\mu < \omega_2$. If this would be the case, then fix any ultrafilter $p \in K_\mu$. Notice that if $B \in CO(X)$ is such that $\{n\colon B \cap H(n, \mu) \neq \emptyset\}$ is in $p \in K_\mu$, then $\{n\colon H(n, \mu) \setminus B \neq \emptyset\}$ is not in p, since otherwise, we could use B as our B_μ. Therefore, there is a unique point, call it y, which is a p-limit of $\{H(n, \mu)\colon n \in \omega\}$.

By our assumption on μ, each $H(n, \mu)$ is a non-empty $G_{|\lambda+\mu|} = G_\lambda$ in $\overline{A}$. Therefore, there is a sequence $\{a(n, \xi)\colon \xi < \lambda\} \subseteq A$ converging to the closed set $H(n, \mu)$. For each $\xi < \lambda$, we are assuming that $\overline{\{a(n, \xi)\colon n \in \omega\}}$ is pseudoradial and hence this set is actually contained in A. For each ξ, let y_ξ be the p-limit of $\{a(n, \xi)\colon n \in \omega\}$. We claim that $\{y_\xi\colon \xi \in \lambda\}$ converges to y. Indeed, if $y \in B \in CO(X)$, then there is a $U \in p$ such that $H(n, \mu) \subseteq B$ for each $n \in U$. Since λ has uncountable cofinality, there is a $\gamma < \lambda$ such that $\{a(n, \xi)\colon \gamma < \xi < \lambda\} \subseteq B$ for each $n \in U$. Therefore, $y_\xi \in B$ for each $\xi \in (\gamma, \lambda)$. This contradicts the radial closedness of A and so we conclude that our construction actually goes all the way to ω_2.

To finish, observe that, by the pressing down lemma, there would be ω_2 many $\mu \in \omega_2$ with the same value for σ_μ and this would result in an independent family of clopen subsets of X of cardinality ω_2. Then, X would map onto 2^{ω_2}. □

Now, with the aid of Theorem 2.1, we obtain the following partial answer to Problem 1.

Corollary 2.2. *Assume that 2^{ω_2} is not pseudoradial. If X is a compact space which is not pseudoradial, then X has a closed separable subspace which is not pseudoradial.*

Proof. Suppose that X is a space which is not pseudoradial. By Theorem 2.1, we may suppose that there are mappings of X onto each of I^{ω_2} and I^{ω_1}. If CH holds then I^{ω_1} is not pseudoradial (and separable), if CH fails, then I^{ω_2} is separable (and is assumed to be not pseudoradial). In either case, X has a compact separable subspace mapping onto a separable non-pseudoradial space. Thus X contains a closed separable non-pseudoradial subspace. □

Corollary 2.2 reinforces Šapirovskiĭ's opinion that the structure of the space 2^{ω_2} is very important in the study of pseudoradial spaces.

This ties in with the following interesting still open problem:

Problem 2. Is it true in ZFC that 2^{ω_2} is not pseudoradial?

We explore the closer connections. Note that Martin's Axiom implies that $\mathfrak{p}$ is $\mathfrak{c}$, hence the assumption that $\mathfrak{p}^+ \geqslant \mathfrak{c}$ is a slightly weaker hypothesis.

Lemma 2.3. *If $\mathfrak{p}^+ \geqslant \mathfrak{c}$ then any radially closed subset of a compact sequentially compact space is ω-closed.*

Proof. Let A be a radially closed subset of the compact sequentially compact space X. If A is not ω-closed, then there is a countable set $B \subseteq A$ such that $\overline{B} \setminus A \neq \emptyset$. Since $\overline{B} \cap A$ is radially closed, according to Theorem 1.2 there exists a closed relatively G_λ subset H of $\overline{B} \setminus A$ such that $\lambda^+ < \mathfrak{c}$. We then have $\lambda < \mathfrak{p}$ and so there exists a sequence $S \subseteq B$ which converges to H. Since X is sequentially compact, the set S has in turn a subsequence converging to a point belonging to H. But this contradicts the radial closedness of A and we conclude that A must be ω-closed. □

An argument similar to that used in the proof of Theorem 2.1 gives:

Theorem 2.4. *If $\mathfrak{p}^+ \geqslant \mathfrak{c}$ then a compact sequentially compact space X which is not pseudoradial maps onto I^{ω_2}.*

Proof. Again we assume that X is zero-dimensional. Suppose that X is compact, sequentially compact and not pseudoradial. Observe first that, because of Theorem 1.1, we must have $\mathfrak{c} > \omega_2$ and so $\mathfrak{i} \geqslant \mathfrak{p} \geqslant \omega_2$. Let A be a subset which is radially closed but not closed and repeat the argument in the proof of Theorem 2.1 for some $H \subseteq \overline{A} \setminus A$. Since we want to show that X maps onto 2^{ω_2}, we must check that the construction made in that proof cannot stop at some stage $\mu < \omega_2$. For this, we may argue exactly as in Theorem 2.1, taking into account that because of Lemma 2.3 our set A is ω-closed. □

Of course Theorem 2.4 may be reformulated, for easier referencing, in the following form.

Corollary 2.5. *If $\mathfrak{p}^+ \geqslant \mathfrak{c}$ then a compact sequentially compact space which cannot be mapped onto I^{ω_2} is pseudoradial.*

No compact sequentially compact space maps onto $I^{\mathfrak{c}}$, hence Theorem 1.1 follows from Corollary 2.5. Corollary 2.5 also covers the case $\mathfrak{p} = \mathfrak{s} = \omega_2$ and $\mathfrak{c} = \omega_3$ but leaves open the following.

Problem 3. Does $\mathfrak{c} \leqslant \mathfrak{p}^+ \leqslant \omega_3$ imply that any compact sequentially compact space is pseudoradial?

A particular case of Corollary 2.5 is in the following:

Corollary 2.6. *If $\mathfrak{p}^+ \geqslant \mathfrak{c}$ then any compact sequentially compact space with tightness not exceeding ω_1 is pseudoradial.*

The above result improves Theorem 8 of [5], relative only to the case of countable tightness.

Theorem 2.4 also says something about the productivity of the class of pseudoradial spaces.

Corollary 2.7. *If $\mathfrak{p}^+ \geqslant \mathfrak{c}$ and 2^{ω_2} is not pseudoradial, then the product of countably many compact pseudoradial spaces is again pseudoradial.*

Proof. Suppose that X is a countable product of compact pseudoradial spaces. By Theorem 2.4, it suffices to show that X does not map onto I^{ω_2}. We know that no factor of X does. Then, it is enough to apply a result of Šapirovskiĭ [7, Theorem 5], namely if a product of fewer than κ compact spaces maps onto I^{κ} (for a regular cardinal κ) then one factor must map onto I^{κ}. □

We will show (Proposition 2.10) that the assumption about 2^{ω_2} can be omitted in the case when $\mathfrak{c} = \omega_3$. It can also be omitted if we restrict the class of spaces.

Corollary 2.8. *If $\mathfrak{p}^+ \geqslant \mathfrak{c}$ then the class of compact pseudoradial spaces of weight not exceeding ω_2 is countably productive.*

Proof. Let X be a countable product of compact pseudoradial spaces of weight not exceeding ω_2. If 2^{ω_2} is not pseudoradial then just apply Corollary 2.7. In the other case, X can be embedded into the pseudoradial space I^{ω_2} as a closed subspace. □

Here is an attempt towards understanding ω_1 convergence in pseudoradial spaces. This result will be useful in examining products.

Lemma 2.9. *If Z is a compact pseudoradial space and $\{z_\alpha\colon \alpha \in \omega_1\}$ is an ω_1-sequence, then there is a cub C contained in ω_1 and a converging sequence $\{z'_\gamma\colon \gamma \in C\}$ such that for each $\beta < \gamma \in C$, z'_γ is in the closure of $\{z_\alpha\colon \beta < \alpha < \gamma\}$.*

Proof. We consider two cases. First case is that for each $\beta < \omega_1$, the union of the increasing sequence of closed sets

$$\bigcup\left\{\overline{\{z_\xi\colon \beta < \xi < \alpha\}}\colon \alpha \in \omega_1\right\}$$

is closed. In this case, fix any complete accumulation point y of $\{z_\alpha\colon \alpha \in \omega_1\}$. Define $f:\omega_1 \to \omega_1$ in such a way that $y \in \overline{\{z_\alpha\colon \beta < \alpha < f(\beta)\}}$ for any β and take a cub $C \subseteq \omega_1$ so that if $\beta < \gamma \in C$ then $f(\beta) < \gamma$. For every $\gamma \in C$, z'_γ is taken to be this complete accumulation point y.

Second case is that there is a β (taken to be 0 for simplicity) such that the above union is not closed. Let $Z_\alpha = \overline{\{z_\xi\colon \xi < \alpha\}}$ and let $\{y_\alpha\colon \alpha < \omega_1\}$ be a sequence contained in $\bigcup\{Z_\alpha\colon \alpha < \omega_1\}$ which converges to a point outside this union. For any α denote by β_α the smallest ordinal such that $y_\alpha \in Z_{\beta_\alpha}$. Let $A = \{\beta_\alpha\colon \alpha < \omega_1\}$ and $z'_{\beta_\alpha} = y_\alpha$. Finally, let C be the set of limit points of A in ω_1 and for any $\gamma \in C \setminus A$ let z'_γ be a point belonging to $\bigcap\{\overline{\{z'_\xi\colon \xi \in (\gamma \setminus \delta) \cap A\}}\colon \delta < \gamma\}$. The sequence $\{z'_\gamma\colon \gamma \in C\}$ has the required properties. □

Proposition 2.10. *If $\mathfrak{p}^+ \geqslant \mathfrak{c} = \omega_3$ then the class of compact pseudoradial spaces is countably productive.*

Proof. Let $\{X_i\colon i \in \omega\}$ be a family of compact pseudoradial spaces and assume, for a contradiction, that $X = \prod_{i\in\omega} X_i$ is not pseudoradial. Let A be a radially closed non-closed subset of X and let H be a closed G_λ subset of $\overline{A} \setminus A$, with λ minimal. Because of Theorem 1.2, we have $\lambda = \omega_1$. Furthermore, Lemma 2.3 implies that A is ω-closed. Let $S = \{x_\alpha\colon \alpha < \omega_1\} \subseteq A$ be a sequence converging to H. By applying Lemma 2.9 to the sequence $\pi_0[S]$, we obtain a cub $C_0 \subseteq \omega_1$ and a sequence $\{x_{0,\alpha}\colon \alpha \in C_0\} \subseteq X_0$ converging to a point x'_0, in such a way that $x_{0,\alpha}$ is in the closure of every final segment of the sequence $\{\pi_0(x_\beta)\colon \beta < \alpha\}$. Let U_α denote an ultrafilter on α so that $x_{0,\alpha}$ is the U_α-limit of the α-sequence $\{\pi_0(x_\beta)\colon \beta < \alpha\}$, and every final segment of α is a member of U_α. Now we use that A is ω-closed to deduce that for each $\alpha \in C_0$, the U_α-limit of the sequence $\{x_\beta\colon \beta < \alpha\}$ is a member of A. Of course this limit is equal to $(x_{0,\alpha}, y_\alpha)$ for some $y_\alpha \in \prod_{1\leqslant i<\omega} X_i$. Let $S_0 = \{(x_{0,\alpha}, y_\alpha)\colon \alpha \in C_0\}$ and apply the same argument to the sequence $\pi_1[S_0]$. We get a cub $C_1 \subseteq C_0$ and a sequence $\{x_{1,\alpha}\colon \alpha \in C_1\} \subseteq X_1$ converging to x'_1. Then choose points $z_\alpha \in \prod_{i\in\omega\setminus\{1\}} X_i$ so that $(x_{1,\alpha}, z_\alpha) \in A$ and call S_1 the sequence so obtained for α ranging in C_1. It is clear that $\pi_{0,1}[S_1]$ converges to (x'_0, x'_1). Continuing we obtain a decreasing sequence, $\{C_i\colon i \in \omega\}$, of cub's and sequences

$$S_i = \left\{s_{i,\alpha} = (x_{0,\alpha}, x_{1,\alpha}, \ldots, x_{i,\alpha}, z_{i,\alpha})\colon \alpha \in C_i\right\} \subset A$$

(where each $x_{j,\alpha} \in X_j$) so that $\{(x_{0,\alpha}, \ldots, x_{i,\alpha})\colon \alpha \in C_i\}$ converges to $(x'_0, \ldots, x'_i)$, and, for each $\beta < \alpha \in C_i$, $s_{i,\alpha}$ is a limit of $\{x_\xi\colon \beta < \xi < \alpha\}$.

To finish, set $C = \bigcap_{i\in\omega} C_i$ and let $x' \in X$ be the point whose ith coordinate is x'_i. Also, for each $\alpha \in C$ let $x'_\alpha = (x_{0,\alpha}, x_{1,\alpha}, \ldots)$. Since x'_α is the limit point of the converging sequence $\{s_{i,\alpha}\colon i \in \omega\} \subset A$, x'_α is in A. The sequence $\{x'_\alpha\colon \alpha \in C\}$ converges to x' and x', being a complete accumulation point of $\{x_\alpha\colon \alpha < \omega_1\}$, belongs to $H \subset \overline{A} \setminus A$. Then we reach a contradiction. □

Proposition 2.11 $[\mathfrak{p}^+ \geqslant \mathfrak{c}]$. *If X and Y are compact pseudoradial spaces and $R\chi(X) \leqslant \omega_1$ then $X \times Y$ is pseudoradial.*

Proof. By contradiction, let A be a radially closed non-closed subset of $X \times Y$. Since every set of the form $\{x\} \times Y \cap A$ is closed, by passing to a relatively closed subset if necessary, we may assume that $\pi_X[A]$ is not closed and so there is a sequence $S \subseteq A$ such that $\pi_X[S]$ converges to some point outside $\pi_X[A]$. If S is countable, then the sequential compactness of $X \times Y$ leads to a contradiction with the fact that A is radially closed. So assume $|S| = \omega_1$ and let $S = \{(x_\alpha, y_\alpha)\colon \alpha < \omega_1\}$. Because of Lemma 2.3, we have that A is ω-closed. Now, let us apply Lemma 2.9 to the sequence $\pi_Y[S] = \{y_\alpha\colon \alpha < \omega_1\}$. Fix any cub $C \subseteq \omega_1$ and a sequence y'_γ for $\gamma \in C$ so that y'_γ is in the closure of every final segment of $\{y_\alpha\colon \alpha < \gamma\}$ and so that this sequence converges. Let U_γ denote an ultrafilter on γ so that y'_γ is the U_γ-limit of the γ-sequence $\{y_\alpha\colon \alpha < \gamma\}$. Of course we assume that every final segment of γ is a member of U_γ. Now we use that A is ω-closed to deduce that for each $\gamma \in C$, the U_γ-limit of the sequence $\{(x_\alpha, y_\alpha)\colon \alpha < \gamma\}$ is a member of A. Of course this limit is equal to (x'_γ, y'_γ) for some x'_γ. The sequence $S' = \{(x'_\gamma, y'_\gamma)\colon \gamma \in C\}$ converges and, since $\pi_X[S]$ and $\pi_X[S']$ have the same limit point, we see that the limit point of S' is outside A—again a contradiction with the radial closeness of A. $\square$

Since the sequential closure is, in general, smaller than the topological closure, we may consider the following strengthening of Lemma 2.9:

Statement 2.9′. *Let Z be a compact pseudoradial space and $\{z_\alpha\colon \alpha < \omega_1\} \subset Z$ a set of size ω_1. Then there is a sequence of points $\{x_\alpha\colon \alpha \in C\}$, for some cub set $C \subset \omega_1$, such that, for each α, x_α is in the sequential closure of the final segment $\{z_\beta\colon \alpha < \beta < \omega_1\}$ and $\{x_\alpha\colon \alpha \in C\}$ converges.*

One can check Lemma 2.9 follows from Statement 2.9′ as follows. Let C and $\{x_\alpha\colon \alpha \in C\}$ be as given in Statement 2.9′. By passing to a subset of C, we can assume that for each $\alpha < \gamma$ both in C, x_α is in the closure of $\{z_\beta\colon \alpha < \beta < \gamma\}$. Let C' denote the set of limits of C and for each $\gamma \in C'$, let z'_γ be any point that, for each $\beta < \gamma$, is a limit of $\{x_\alpha\colon \beta < \alpha < \gamma\}$. The fact that $\{z'_\gamma\colon \gamma \in C'\}$ converges follows easily from the assumption that $\{x_\alpha\colon \alpha \in C\}$ converges.

Using Statement 2.9′ and the same argument as in Proposition 2.10, we could actually prove that $\mathfrak{c} \leqslant \omega_3$ suffices to guarantee countable productivity of the class of compact pseudoradial spaces.

At the moment, we only know that Statement 2.9′ is provable from $\mathfrak{c} \leqslant \omega_2$, but this adds nothing to our knowledge of products.

3. On some questions of Šapirovskiĭ

The material in this section was inspired by the last paper of Šapirovskiĭ [8], written by Nyikos and Vaughan after the workshop at Madison in June 1991. The main point of that paper was the result that $\mathfrak{c} = \omega_1$ implies that every compact sequentially compact space is pseudoradial. To prove this theorem, a major role is played by the notion of $\aleph_0$-pseudoradialness. This fits well in our theme.

A space X is $\aleph_0$-pseudoradial provided that every ω-closed non-closed set $A \subseteq X$ contains a sequence converging to a point outside A. Šapirovskiĭ proved [8, Corollary 3.1] that a compact space which cannot be mapped onto I^{ω_1} is $\aleph_0$-pseudoradial and then he formulated the conjecture that this result still holds by replacing I^{ω_1} with I^{ω_2}.

Following the same argument in the proof of our Theorem 2.1, just starting by taking a radially closed ω-closed non-closed set A, we may actually prove:

Theorem 3.1. *Assume $\mathfrak{i} > \omega_1$. If the compact space X cannot be mapped onto I^{ω_2} then X is $\aleph_0$-pseudoradial.*

Another question in [8, Problem I] was whether in ZFC it is true that I^{ω_2} is not $\aleph_0$-pseudoradial (observe that this is the case, for instance, in any model in which the cub filter on ω_1 has character ω_2).

Also the above question is a weaker version of the more fundamental Problem 2, mentioned in the previous section.

We show that the answer to this question of Šapirovskiĭ is in the negative.

Recall [6] that P_1 is the principle that if $\mathcal{F}$ is a family of fewer than 2^{ω_1} subsets of ω_1 and if each countable intersection of members of $\mathcal{F}$ is uncountable, then there is an uncountable set $Y \subseteq \omega_1$ such that $Y \setminus F$ is countable for each member F of $\mathcal{F}$.

It is shown in [6, Theorem 7.13, p. 286] that there is a model of ZFC satisfying $P_1 + \mathfrak{c} = \omega_1 + 2^{\omega_1} > \omega_2$.

Theorem 3.2 $[P_1 + \mathfrak{c} = \omega_1 + 2^{\omega_1} > \omega_2]$. *$I^{\omega_2}$ is $\aleph_0$-pseudoradial.*

Proof. Let $A \subseteq I^{\omega_2}$ be a ω-closed non-closed set. If A were ω_1-closed then, taking into account that the space I^{ω_2} has character ω_2, we could easily prove that any point of $\overline{A} \setminus A$ is the limit of a ω_2-sequence contained in A. Thus, we are reduced to consider the case that A has a dense subset B of size ω_1. There exists a countably compact set $C \subseteq A$ such that $B \subseteq C$ and $|C| = \mathfrak{c} = \omega_1$. Write $C = \omega_1 \subseteq A$. If $\mathcal{F}$ is the trace on C of a local base at some point $x \in \overline{C} \setminus A$, then the intersection of any countable subfamily of $\mathcal{F}$ has size ω_1. To see this, take $\{F_n = U_n \cap C\colon n \in \omega\} \subseteq \mathcal{F}$ and fix $\gamma \in \omega_1$. Since $\overline{[0, \gamma]} \subseteq A$, we may fix a neighborhood V on x such that $\overline{V} \cap [0, \gamma] = \emptyset$. Select for any n a point $\alpha_n \in V \cap W_0 \cap \cdots \cap W_n \cap C$, where W_n is open and $x \in W_n \subseteq \overline{W_n} \subseteq U_n$. Select $\alpha_\gamma \in C$ to be a limit point of $\{\alpha_n\colon n \in \omega\}$. Clearly $\alpha_\gamma \in \bigcap\{F_n\colon n \in \omega\}$ and $\gamma < \alpha_\gamma$. To finish, by applying P_1, we see that C has a subsequence converging to x. $\square$

Problem 4. Are there models in which the Continuum Hypothesis fails and in which I^{ω_2} is $\aleph_0$-pseudoradial? (E.g., simply add Cohen reals to the model mentioned above.)

References

[1] A. Bella, A. Dow, On R-monolithic spaces, Topology Appl. 105 (2000) 89–97.

[2] E.K. van Douwen, The integers and topology, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretc Topology, North-Holland, Amsterdam, 1984, pp. 111–167.

[3] A. Dow, On compact separable radial spaces, Canad. Math. Bull. 40 (1997) 422–432.

[4] A. Dow, I. Juhász, L. Soukup, Z. Szentmiklóssy, More on sequentially compact implying pseudoradial, Topology Appl. 73 (1996) 191–195.

[5] I. Juhász, Z. Szentmiklóssy, Sequential compactness versus pseudoradiality in compact spaces, Topology Appl. 50 (1993) 47–53.

[6] K. Kunen, Set Theory, An Introduction to Independence Results, Elsevier Science Publishers B.V., Amsterdam, 1980.

[7] B. Šapirovskiĭ, Maps onto Tychonoff cubes, Russian Math. Surveys 35 (3) (1980) 145–156.

[8] B. Šapirovskiĭ, The equivalence of sequential compactness and pseudoradialness in the class of compact T_2 spaces, assuming CH, Ann. New York Acad. Sci. 704 (1993) 322–327.

[9] J. Vaughan, Small uncountable cardinals, in: J. van Mill, M. Reed (Eds.), Open Problems in Topology, Elsevier Science Publishers B.V., Amsterdam, 1990, pp. 195–216.